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Note

Higher Order Two-Point Quasi-Fractional Approximations to the Bessel Functions $J_0(x)$ and $J_1(x)$

I. INTRODUCTION

In a recent paper [1] we presented a method to obtain quasi-fractional approximations which improves a previously published one [2, 3]. Mainly, in that method all the parameters of the approximation are real and straightforwardly determined and the accuracy is increased. This method is based on the simultaneous use of expansions for small and large values of the independent variable. The first method was described in relation to $J_0(x)$ and for the lowest order of approximation. Here, we extend our results in order to include $J_1(x)$ and higher order approximations to both $J_0(x)$ and $J_1(x)$.

As in the two-point Padé method, we use simultaneous expansions of a given function around zero and infinity. However, unlike the various ways of using the two-point Padé method [4–7], which use purely rational functions, our approximations in order to reproduce the essential singularity at infinity have a different structure combining a rational function with exponentials and fractional-order powers; this is why we call them quasi-fractional. We use the asymptotic expansion instead of the Laurent series at infinity.

The resulting formulae for $J_0(x)$ and $J_1(x)$ are approximations valid for the full range $x \geqslant 0$. Other algorithms break up the range and use a different approximation in each region [8–13].

II. OUTLINE OF THE METHOD

We want fo find approximate solutions to the Bessel Functions $J_\nu(x)$ $(\nu = 0, 1)$ valid for small and large values of x. If a fractional solution, $y = (\sum p_i x^i)(\sum q_j x^j)^{-1}$, is substituted in the usual Bessel equation and we try to find the parameters p_i, q_j through the highest power of the variable, we get $p_n q_n = 0$. The latter would mean either p_n or q_n equal to zero and this would lead to trivial solutions. To bypass this difficulty a term $(1+x)^{-1/2}$ is factored out of the solution and auxiliary functions are used which are defined as

$$\nu = 0, 1;$$

$$\begin{aligned} J_\nu(x) &= (1+x)^{-1/2}\,[w(x)\exp(ix) + w^*(x)\exp(-ix)] \\ &= (1+x)^{-1/2}\,[u(x)\cos x + v(x)\sin x] \end{aligned} \tag{1}$$

$$w(x) \equiv \tfrac{1}{2}[u(x) - iv(x)], \tag{2}$$

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where $w(x)$ satisfies the equation

$$x^2(1+x)^2 w''(x) + x(1+x)(1+2ix+2ix^2) w'(x) + [-\nu^2 + (-\tfrac{1}{2}+i-2\nu^2) x + (\tfrac{1}{4}+i-\nu^2) x^2] w(x) = 0. \tag{3}$$

This differential equation is now suitable for direct substitution of a fractional solution since the highest degree equation in x (after rationalizing) leads to a non-trivial solution of the coefficients.

The power series of the function $(1+x)^{1/2} J_\nu(x)$ is

$$(1+x)^{1/2} J_\nu(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{4}$$

where

$$a_k = \frac{1}{2^\nu} \sum_{j=0}^{(k/2)} \binom{\nu+\frac{1}{2}}{k-2j} \frac{(-1)^j}{2^{2j} j!\, \Gamma(j+\nu+1)}. \tag{5}$$

On the other hand, the differential equation (3) and the knowledge of the leading terms of the asymptotic expansions of $J_\nu(x)$, enable us to find the full asymptotic expansion for $w(x)$ whereby the corresponding asymptotic expansions for $u(x)$ and $v(x)$ can be found

$$u(x) = \sum_{k=0}^{\infty} B_k x^{-k}, \qquad v(x) = \sum_{k=0}^{\infty} b_k x^{-k}, \tag{6a}$$

where

$$B_k = 2\,\mathrm{Re}(\beta_k), \qquad b_k = -2\,\mathrm{Im}(\beta_k) \tag{6b}$$

$$\beta_k = \frac{(-i)^\nu}{\sqrt{\pi}\,(1+i)} \left\{ \binom{\frac{1}{2}}{k} + \sum_{j=1}^{k} \binom{\frac{1}{2}}{k-j} \frac{(-i)^j}{2^j j!} \prod_{l=1}^{j} [l(l-1) + \tfrac{1}{4} - \nu^2] \right\}. \tag{6c}$$

Now, let the functions $u(x)$ and $v(x)$ be approximated by polynomial quotients of degree n, thus

$$\hat{u}(x) = \frac{\sum_{i=0}^{n} P_i x^i}{\sum_{j=0}^{n} q_j x^j}, \qquad \hat{v}(x) = \frac{\sum_{i=0}^{n} p_i x^i}{\sum_{j=0}^{n} q_j x^j} \tag{7}$$

in which we have chosen identical denominators to get linear equations for the parameters.

Hence, we try n-order fractional approximations of the type

$$\hat{J}_\nu(x) = \frac{1}{\sqrt{1+x}} \left(\frac{\sum_{i=0}^{n} P_i x^i}{\sum_{j=0}^{n} q_j x^j} \cos x + \frac{\sum_{i=0}^{n} p_i x^i}{\sum_{j=0}^{n} q_j x^j} \sin x \right), \tag{8}$$

$\nu = 0, 1$.

We choose $q_0 = 1$. The remaining $3n+2$ parameters are determined from the coefficients a_k, B_k, and b_k of the ascending series (5) and the two asymptotic expansions (6) as

$$\left(\sum_{j=0}^{n} q_j x^j\right)\left(\sum_{k=0}^{\infty} a_k x^k\right) = \left(\sum_{i=0}^{n} P_i x^i\right)\left(\sum_{x=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}\right) + \left(\sum_{i=0}^{n} p_i x^i\right)\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right) \tag{9}$$

$$\left(\sum_{j=0}^{n} q_{n-j} x^{-j}\right)\left(\sum_{k=0}^{\infty} B_k x^{-k}\right) = \sum_{i=0}^{n} P_{n-i} x^{-i} \tag{10a}$$

$$\left(\sum_{j=0}^{n} q_{n-j} x^{-j}\right)\left(\sum_{k=0}^{\infty} b_k x^{-k}\right) = \sum_{i=0}^{n} p_{n-i} x^{-i}. \tag{10b}$$

By equating the lowest order coefficients in the above equations, a set of linear equations in P_i, p_i, and q_j is obtained. Let m be the number of equations arising from equating coefficients in (9), and let r and s be the analogous numbers for the equations coming from (10a) and (10b), respectively. In order to get a compatible system of equations it is necessary that $3n+2$, the number of remaining unknowns, be equal to $m+r+s$.

TABLE I

Approximation Parameters for $\hat{J}_0(x)$ Corresponding to Different Orders n of Approximation together with the Maximum Value of the Error $|\widehat{\Delta J}_0| \equiv |\hat{J}_0 - J_0|$ for Each n

$n=1$			
$m=3$, $r=1$, $s=1$	$p_0 = 1.00$	$p_0 = 1.35$	$q_0 = 1.00$
$\lvert\widehat{\Delta J}_{\max}\rvert = 3.5 \times 10^{-3}$ at $x = 3.70$	$p_1 = 1.10$	$p_1 = 1.10$	$q_1 = 1.95$
$n=2$			
$m=2$, $r=3$, $s=3$	$p_0 = 1.000$	$p_0 = 3.550$	$q_0 = 1.000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 3.9 \times 10^{-3}$ at $x = 0.58$	$p_1 = 9.406$	$p_1 = 10.99$	$q_1 = 12.46$
	$p_2 = 6.343$	$p_2 = 6.343$	$q_2 = 11.24$
$n=3$			
$m=6$, $r=2$, $s=3$	$p_0 = 1.0000$	$p_0 = 3.0211$	$q_0 = 1.0000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 1.3 \times 10^{-4}$ at $x = 5.00$	$p_1 = 7.8384$	$p_1 = 9.4658$	$q_1 = 10.359$
	$p_2 = 6.6696$	$p_2 = 7.0423$	$q_2 = 10.831$
	$p_3 = 1.4908$	$p_3 = 1.4908$	$q_3 = 2.6424$
$n=4$			
$m=7$, $r=4$, $s=3$	$p_0 = 1.000000$	$p_0 = 2.001837$	$q_0 = 1.000000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 2.8 \times 10^{-6}$ at $x = 2.40$	$p_1 = 4.637270$	$p_1 = 5.929565$	$q_1 = 6.139108$
	$p_2 = 5.057731$	$p_2 = 5.598357$	$q_2 = 7.792742$
	$p_3 = 2.120459$	$p_3 = 2.204553$	$q_3 = 3.534836$
	$p_4 = 0.3363767$	$p_4 = 0.3363767$	$q_4 = 0.5962123$

III. Results

The values of the best approximation parameters that we obtained for $J_0(x)$ and $J_1(x)$ are given in Tables I and II, respectively. Every set of parameter values is headed by the corresponding order, n, of the approximation.

The figures show the differences between our approximations and the exact functions as tabulated [12]. In Fig. 1 a plot of ten times $\widehat{\Delta J}_0(x)$, defined as $\hat{J}_0(x) - J_0(x)$, is presented for the second degree of the approximation (i.e., $n = 2$). The case $n = 1$ has already been reported in a previous paper [1]. $J_0(x)$ for $n = 3$ and $n = 4$ is also shown. Figure 2 is analogous to Fig. 1 for $\widehat{\Delta J}_1$. Plots of $\widehat{\Delta J}_1$ for $n = 3$ and $n = 5$ are also presented.

TABLE II

Approximation Parameters for $\hat{J}_1(x)$ Corresponding to Different Orders n of the Approximation together with the Maximum Value of the Error $|\widehat{\Delta J}_1| \equiv |\hat{J}_1 - J_1|$ for Each n

$n = 1$			
$m = 3,\ r = 1,\ s = 1$	$p_0 = 0.0$	$p_0 = 2.7$	$q_0 = 1.0$
$\lvert\widehat{\Delta J}_{\max}\rvert = 3.4 \times 10^{-2}$ at $x = 2.34$	$p_1 = -2.2$	$p_1 = 2.2$	$q_1 = 3.9$
$n = 2$			
$m = 5,\ r = 1,\ s = 2$	$p_0 = 0.000$	$p_0 = 8.009$	$q_0 = 1.000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 7.7 \times 10^{-4}$ ar $x = 4.70$	$p_1 = -7.509$	$p_1 = 10.51$	$q_1 = 12.10$
	$p_2 = -4.211$	$p_2 = 4.211$	$q_2 = 7.464$
$n = 3$			
$m = 5,\ r = 3,\ s = 3$	$p_0 = 0.0000$	$p_0 = 35.491$	$q_0 = 1.0000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 3.2 \times 10^{-4}$ at $x = 2.70$	$p_1 = -34.991$	$p_1 = 59.995$	$q_1 = 51.473$
	$p_2 = -31.008$	$p_2 = 35.668$	$q_2 = 53.584$
	$p_3 = -6.2133$	$p_3 = 6.2133$	$q_3 = 11.013$
$n = 4$			
$m = 5,\ r = 5,\ s = 4$	$p_0 = 0.0000$	$p_0 = 115.04$	$q_0 = 1.0000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 1.2 \times 10^{-4}$ at $x = 2.50$	$p_1 = -114.54$	$p_1 = 217.04$	$q_1 = 188.10$
	$p_2 = -122.74$	$p_2 = 154.52$	$q_2 = 210.46$
	$p_3 = -40.482$	$p_3 = 44.251$	$q_3 = 70.639$
	$p_4 = -5.0249$	$p_4 = 5.0249$	$q_4 = 8.9065$
$n = 5$			
$m = 10,\ r = 4,\ s = 3$	$p_0 = 0.000000$	$p_0 = 22.42540$	$q_0 = 1.000000$
$\lvert\widehat{\Delta J}_{\max}\rvert = 5.5 \times 10^{-6}$ at $x = 5.80$	$p_1 = -21.92540$	$p_1 = 62.33123$	$q_1 = 35.36814$
	$p_2 = -44.39716$	$p_2 = 72.38236$	$q_2 = 74.59090$
	$p_3 = -33.59500$	$p_3 = 42.24611$	$q_3 = 57.61484$
	$p_4 = -11.02203$	$p_4 = 12.04760$	$q_4 = 19.23308$
	$p_5 = -1.367421$	$p_5 = 1.367421$	$q_5 = 2.423690$

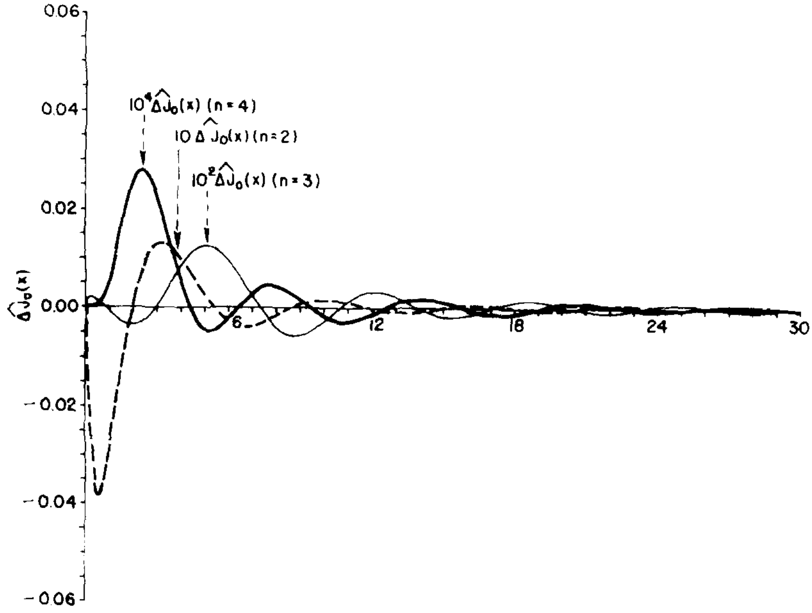

FIG. 1. The error of the approximation $\hat{J}_0(x)$ for the second, third, and fourth orders of approximation multiplied by adequate scale factors.

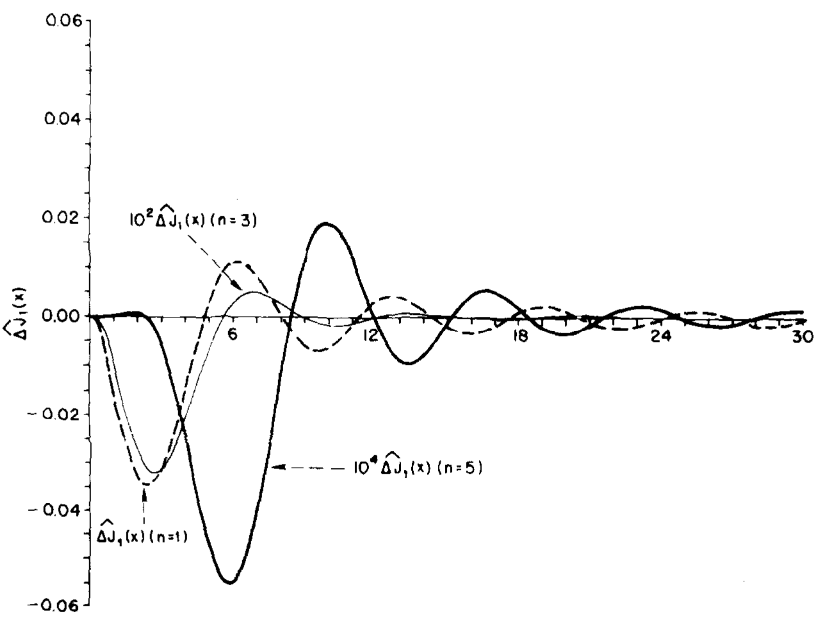

FIG. 2. The error of the approximation $\hat{J}_1(x)$ for the first, third, and fifth orders of approximation multiplied by adequate scale factors.

IV. Summary and Conclusions

Approximations of quasi-fractional type to integer-order Bessel functions have been presented which are valid for the full range $x \geqslant 0$. These are obtained by simultaneous use of expansions near zero and infinity. All the parameters of these approximations are obtained from linear equations with real coefficients, thus yielding real parameters.

The maximum errors for the first degree approximations to $J_0(x)$ and $J_1(x)$ are 3.5×10^{-3} and 3.4×10^{-2}, respectively. The maximum errors of our highest order approximations, $\hat{J}_0(x)$ $(n=4)$ and $\hat{J}_1(x)$ $(n=5)$ are 2.8×10^{-6} and 5.5×10^{-6}, respectively.

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